Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet

Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

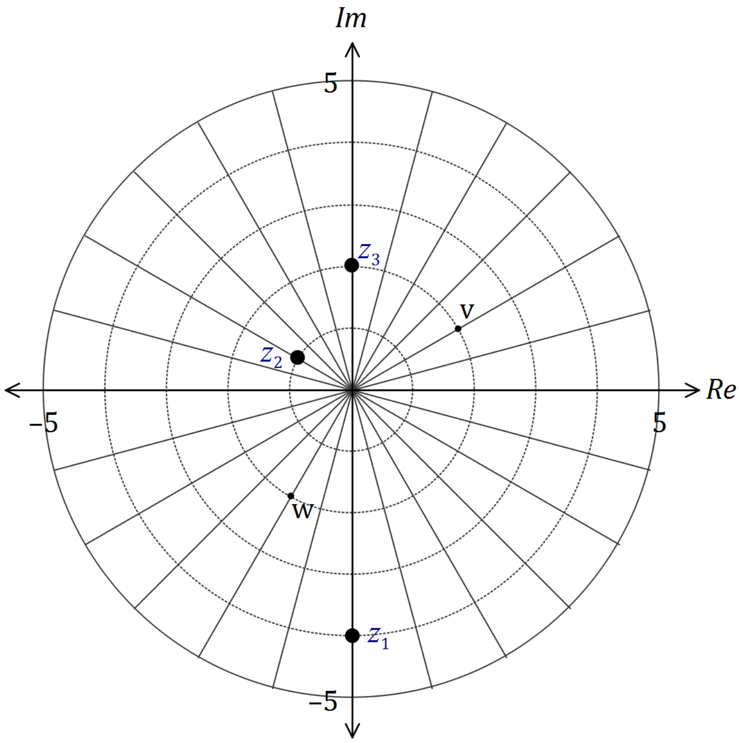
Section Two: Calculator-assumed 65% (98 Marks)

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 (6 marks)

The complex numbers v and w are shown on the Argand diagram below.



On the diagram, clearly mark the complex numbers

(a) $z_1 = vw$. (2 marks)

Solution
Multiply moduli and add arguments
Specific behaviours
✓ correct modulus ✓ correct argument

(b) $z_2 = \dfrac{v}{w}$. (2 marks)

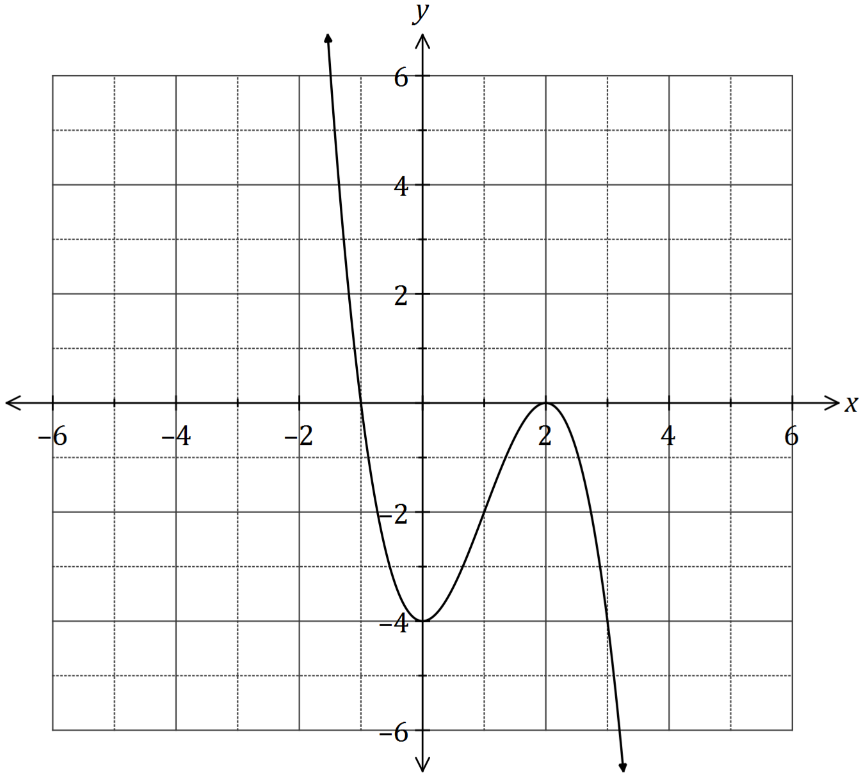
Solution
Divide moduli and subtract arguments
Specific behaviours
✓ correct modulus ✓ correct argument

(c) $z_3 = v - iw$. (2 marks)

Solution
Rotate w 90° anticlockwise and then treat as vector addition
Specific behaviours
✓ correct modulus ✓ correct argument

Question 10 **(8 marks)**

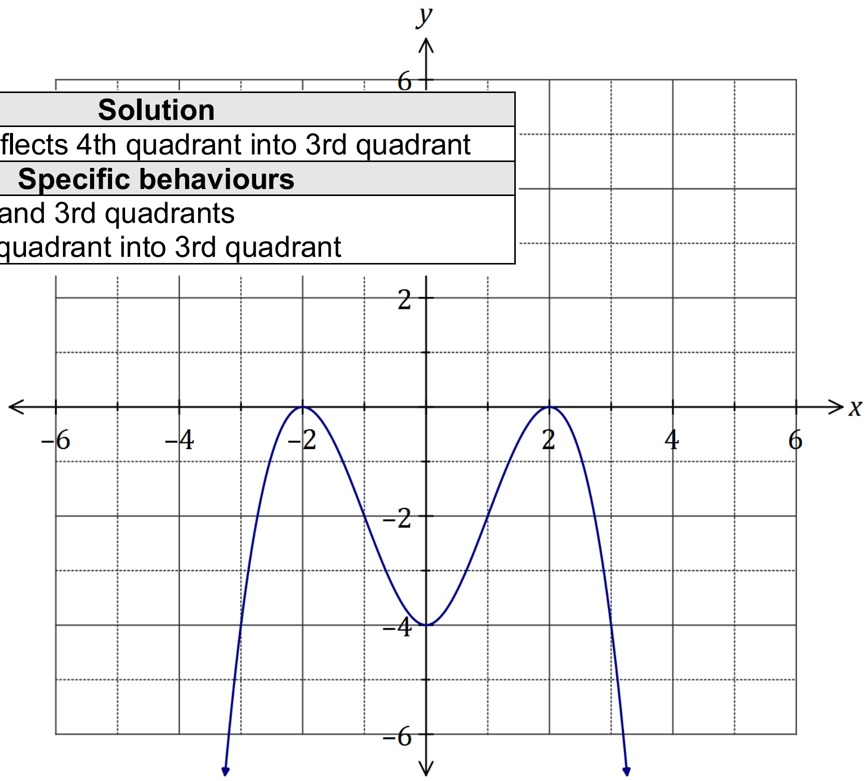
The graph of $y = f(x)$ is drawn below.



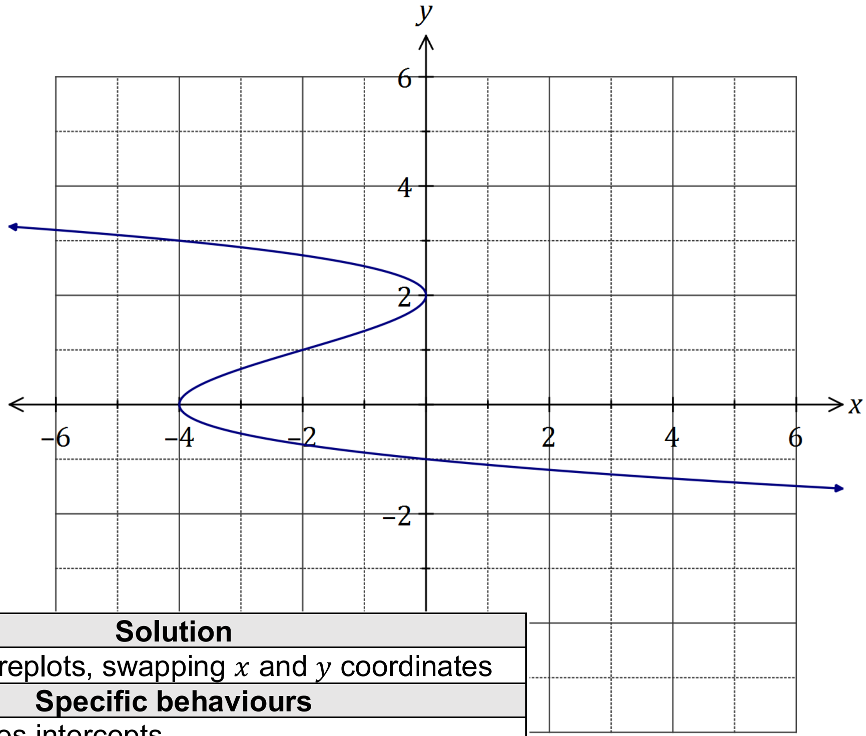
On the axes provided, sketch the graphs of

(a) $y = f(|x|)$. (2 marks)

Solution
See graph - reflects 4th quadrant into 3rd quadrant
Specific behaviours
✓ erases 2nd and 3rd quadrants
✓ reflects 4th quadrant into 3rd quadrant

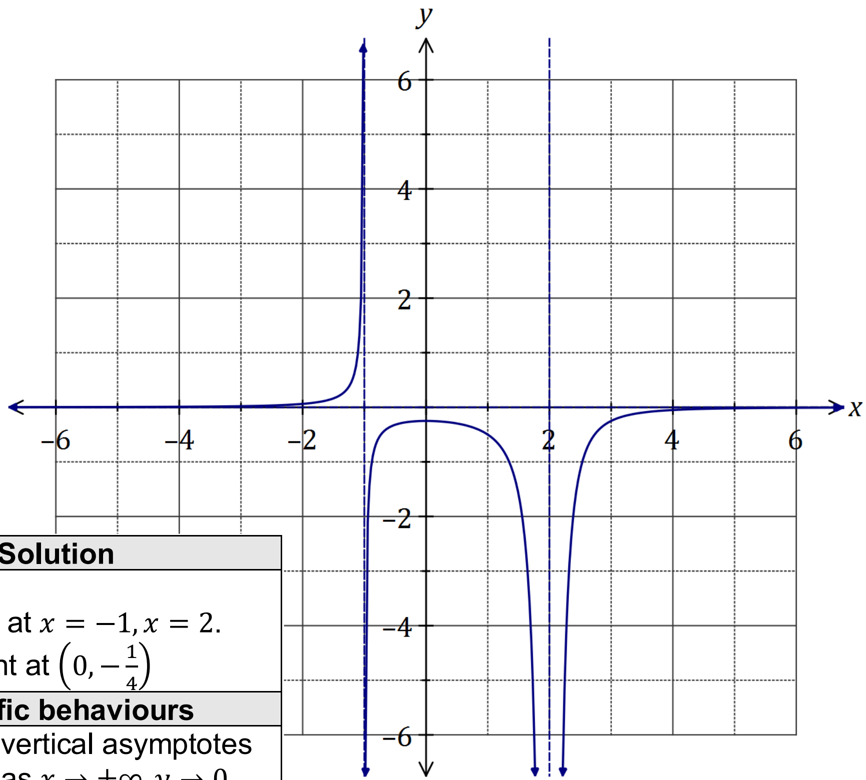


(b) $y = f^{-1}(x)$, the inverse relation. (2 marks)



Solution
See graph - replots, swapping x and y coordinates
Specific behaviours
✓ correct axes intercepts ✓ correct graph

(c) $y = \dfrac{1}{f(x)}$. (4 marks)



Solution
See graph. Asymptotes at $x = -1, x = 2$. Turning point at $\left(0, -\frac{1}{4}\right)$
Specific behaviours
✓ indicates vertical asymptotes ✓ indicates as $x \to \pm\infty, y \to 0$ ✓ indicates turning point ✓ correct graph

Question 11 **(6 marks)**

The position vector $\boldsymbol{r}(t)$ of a model railway train at time t, in an appropriately chosen coordinate system, is given by

$$\boldsymbol{r}(t) = 3\cos kt\,\boldsymbol{i} - 2\sin kt\;\boldsymbol{j}$$

where distances are measured in metres and time is measured in seconds after an appropriately chosen starting time. The number k is positive.

(a) Obtain a Cartesian equation for the path traversed by the train. (3 marks)

Solution $x = 3\cos kt$ and $y = -2\sin kt$, so $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{-2}\right)^2 = \cos^2 kt + \sin^2 kt = 1$ i.e. $\frac{x^2}{9} + \frac{y^2}{4} = 1$	
Marking key/mathematical behaviours	Marks
• obtains the formulae for x and y • uses trig identity • eliminates t correctly	1 1 1

(b) Describe the geometric shape of the path of the train. (1 mark)

Solution The path is an ellipse.	
Marking key/mathematical behaviours	Marks
• obtains correct answer	1

(c) Does the train travel in a clockwise or anticlockwise direction around its closed path? Justify your answer. (2 marks)

Solution $\boldsymbol{r}(0) = 3\,\boldsymbol{i}$ and for t increased such that $kt < \frac{\pi}{2}, \boldsymbol{r}(t)$ is in the fourth quadrant The train is moving in a clockwise direction around the closed path.	
Marking key/mathematical behaviours	Marks
• obtains the correct answer • gives a valid reason	1 1

Question 12 **(9 marks)**

A function is defined by $f(x) = \dfrac{x^2+4x-12}{3x-7}$, $x \neq 0$.

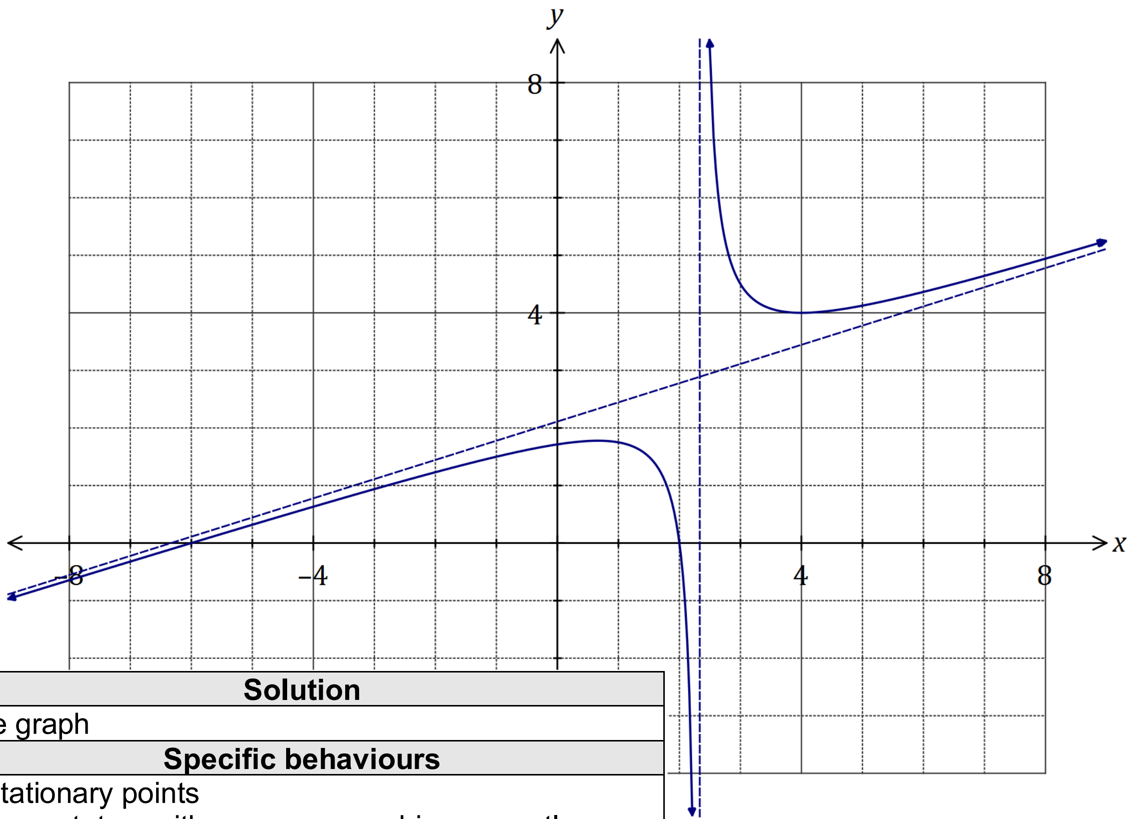
(a) Determine the exact coordinates of all stationary points of the graph of $y = f(x)$. (2 marks)

Solution
$f'(x) = 0 \text{ when } (x-4)(3x-2) = 0 \Rightarrow x = 4, x = \frac{2}{3}$ At $(4, 4)$ and $\left(\frac{2}{3}, \frac{16}{9}\right)$
Specific behaviours
✓ first point ✓ second point

(b) Determine the equation(s) of the asymptote(s) of the graph $y = f(x)$. (3 marks)

Solution
Vertical asymptote: $x = \frac{7}{3}$ $f(x) = \frac{x}{3} + \frac{19}{9} + \frac{25}{9(3x-7)}$ Oblique asymptote: $y = \frac{x}{3} + \frac{19}{9}$
Specific behaviours
✓ vertical asymptote ✓ indicates equivalent form of f ✓ oblique asymptote

(c) Sketch the graph $y = f(x)$ on the axes below. (4 marks)

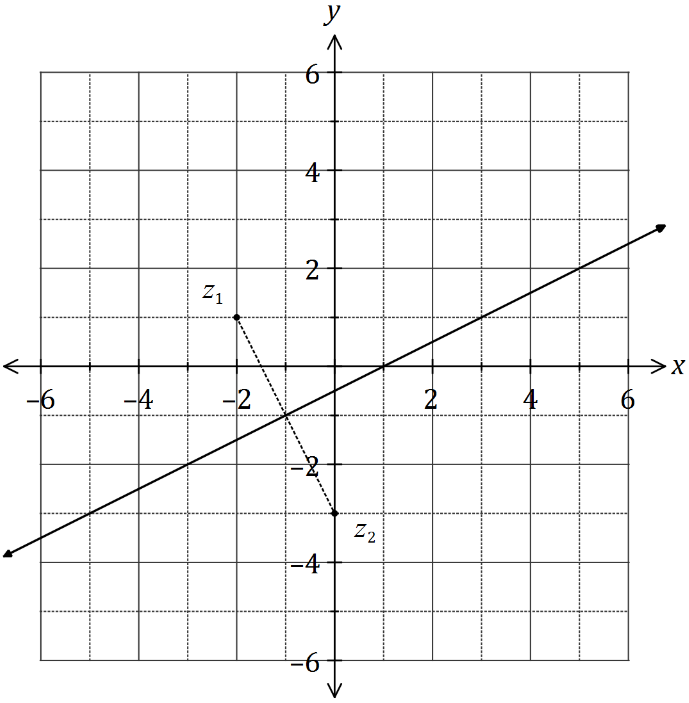


Solution
See graph
Specific behaviours
✓ stationary points ✓ asymptotes, with curve approaching correctly ✓ intercepts ✓ smooth curves

Question 13 **(11 marks)**

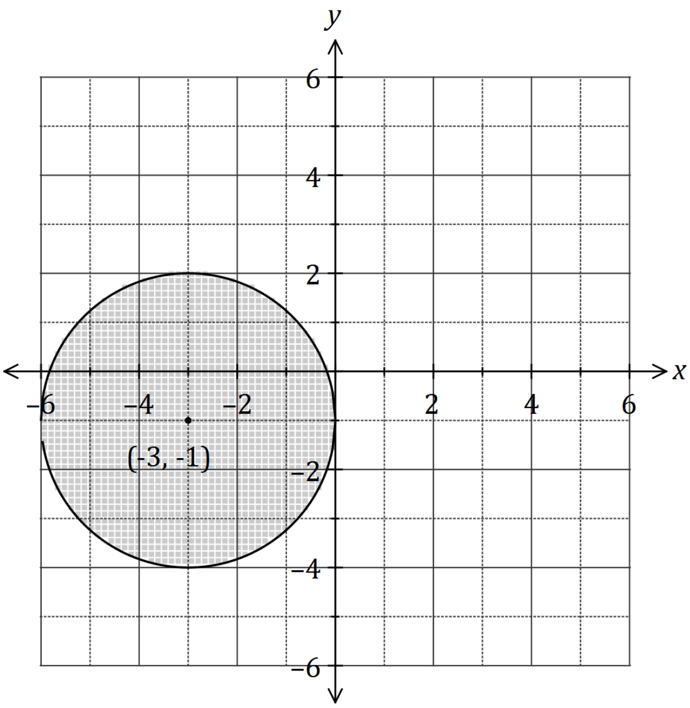
(a) On the Argand planes below, sketch the subsets of the complex plane determined by

(i) $|z + 3i| = |z + 2 - i|$. (3 marks)



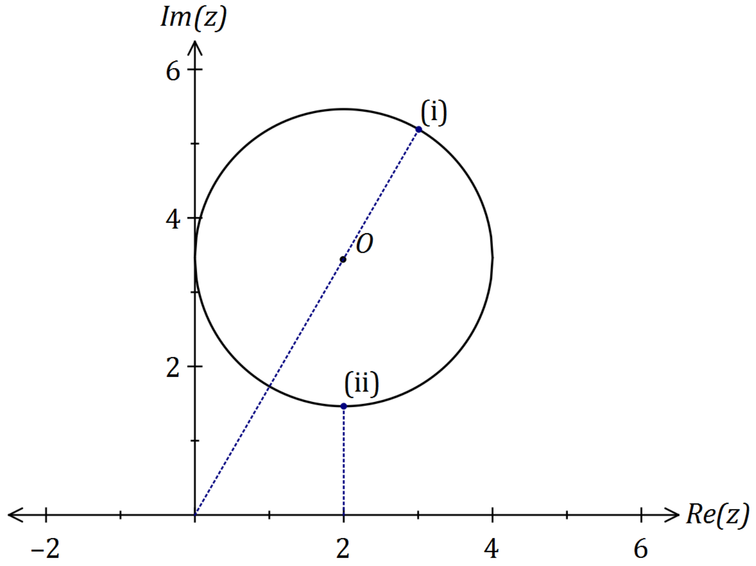
Solution
$\lvert z - (0 - 3i)\rvert = \lvert z - (-2 + i)\rvert$ $y = \frac{1}{2}x - \frac{1}{2}$ See graph
Specific behaviours
✓ indicates use of (0, -3) and (-2, 1) ✓ write equation of straight line ✓ accurate sketch of the line

(ii) $|z + 3 + i| \leq 3$. (3 marks)



Solution
$\lvert z - (-3 - i)\rvert \leq 3$ See graph
Specific behaviours
✓ indicates circle, including boundary ✓ uses correct centre and radius ✓ shades inner region

(b) A subset of the complex plane, a circle with centre O, is shown below.



(i) Mark the position in the plane where $|z|$ is maximised. Label this point (i). (1 mark)

Solution
Maximum when z lies on circumference at greatest distance from origin.
Specific behaviours
✓ indicates location

(ii) Mark the position in the plane where $|z-2|$ is minimised. Label this point (ii). (1 mark)

Solution
Minimum when z lies on circumference at closest point to (2, 0).
Specific behaviours
✓ indicates location

(iii) If the subset shown is $|z-2-2\sqrt{3}i|=2$, determine the maximum and minimum values of $\arg z$. (3 marks)

Solution
Maximum: $\arg z = \frac{\pi}{2}$ Centre: $\arg(2+2\sqrt{3}i) = \frac{\pi}{3}$ Minimum: $\arg z = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$
Specific behaviours
✓ states maximum ✓ indicates argument of centre ✓ uses symmetry to determine minimum

Question 14 **(8 marks)**

The plane P has equation $\mathbf{r} \cdot \mathbf{n} = 11$, where $\mathbf{n} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and the point A has position vector $2\boldsymbol{i} + 5\mathbf{j} - 2\mathbf{k}$.

(a) Determine the Cartesian equation of plane Q that is parallel to P and passes through A. (2 marks)

Solution
$x - y + 2z = (2) - (5) + 2(-2)$ $x - y + 2z = -7$
Specific behaviours
✓ writes LHS of equation ✓ determines constant

(b) Determine the equation of the line L that passes through A and is perpendicular to P. (1 mark)

Solution
$\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
Specific behaviours
✓ writes equation

(c) Determine the position vector of B, the point of intersection of line L with plane P. (3 marks)

Solution
$\begin{pmatrix} 2+\lambda \\ 5-\lambda \\ -2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 11$ $2 + \lambda - 5 + \lambda - 4 + 4\lambda = 11 \Rightarrow \lambda = 3$ $\overrightarrow{OB} = \begin{pmatrix} 2+3 \\ 5-3 \\ -2+2(3) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$
Specific behaviours
✓ substitutes line into plane ✓ solves for parameter ✓ states point of intersection

(d) Determine the exact distance between planes P and Q. (2 marks)

Solution
$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}$ $\lvert\overrightarrow{AB}\rvert = 3\sqrt{6}$
Specific behaviours
✓ determines $\overrightarrow{AB}$ ✓ states distance

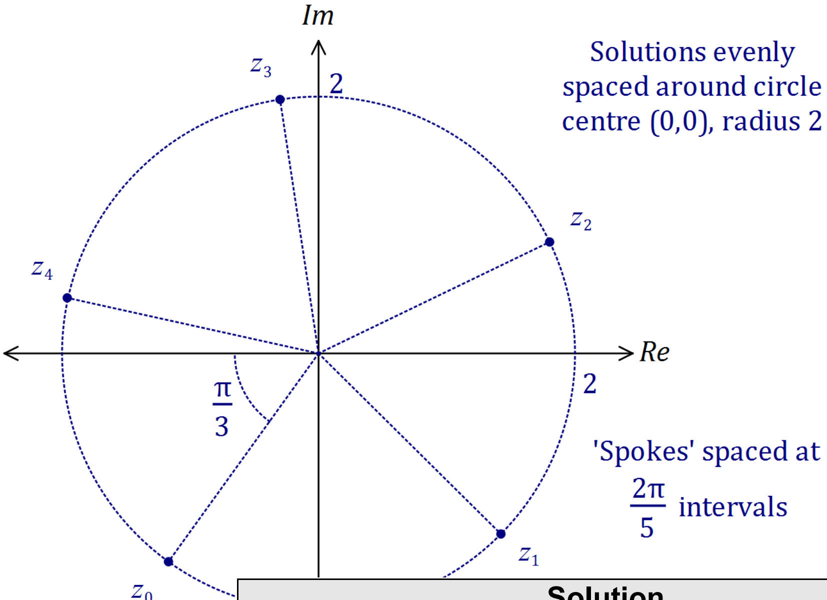
Question 15 **(8 marks)**

Consider the complex equation $z^5 = -16 + 16\sqrt{3}i$.

(a) Solve the equation, giving all solutions in the form $r \text{ cis } \theta$, $r > 0$, $-\pi \le \theta \le \pi$. (4 marks)

Solution
$z^5 = 32 \text{ cis}\left(\frac{2\pi}{3}\right)$ $z = 2 \text{ cis}\left(\frac{2\pi}{15} + \frac{2k\pi}{5}\right), k \in \mathbb{Z}$ $z_0 = 2 \text{ cis}\left(-\frac{2\pi}{3}\right)$ $z_1 = 2 \text{ cis}\left(-\frac{4\pi}{15}\right)$ $z_2 = 2 \text{ cis}\left(\frac{2\pi}{15}\right)$ $z_3 = 2 \text{ cis}\left(\frac{8\pi}{15}\right)$ $z_4 = 2 \text{ cis}\left(\frac{14\pi}{15}\right)$
Specific behaviours
✓ expresses in polar form ✓ uses de Moivre's theorem to obtain general solution ✓ states one correct root ✓ states all roots

(b) Plot the solutions found in part (a) on the Argand diagram below, indicating all key features of the plot. (4 marks)



Solution
See diagram
Specific behaviours
✓ indicates solutions at distance 2 from origin ✓ indicates solutions evenly spaced at $\frac{2\pi}{5}$ intervals ✓ exact argument of one solution shown ✓ indicates approximate position of all five solutions

Question 16 **(12 marks)**

The plane P intersects the axes at the points $A(-3,0,0)$, $B(0,4,0)$ and $C(0,0,1)$.

(a) Demonstrate the use of cross product to find vector $\boldsymbol{n}$ that is normal to the plane P. (2 marks)

Solution $\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AC} = 3\boldsymbol{i} + \boldsymbol{k}$ lie in P so $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (3\boldsymbol{i} + 4\boldsymbol{j}) \times (3\boldsymbol{i} + \boldsymbol{k}) = 4\boldsymbol{i} - 3\boldsymbol{j} - 12\boldsymbol{k}$ is normal to P	
Marking key/mathematical behaviours	Marks
• obtains 2 non-parallel vectors in P • calculates the cross product correctly	1 1

(b) Find a Cartesian equation for P. (2 marks)

Solution Vector equation $\boldsymbol{r} \cdot \boldsymbol{n} = c$ for P is $4x - 3y - 12z = 4 \times -3 = -12$	
Marking key/mathematical behaviours	Marks
• obtains a vector equation • obtains a Cartesian equation	1 1

A vector equation for the line L is $\boldsymbol{r} = (2 + 3\lambda)\boldsymbol{i} + 4\boldsymbol{j} + (1 + \lambda)\boldsymbol{k}$.

(c) Demonstrate the use of dot product to show that line L does not intersect plane P. (3 marks)

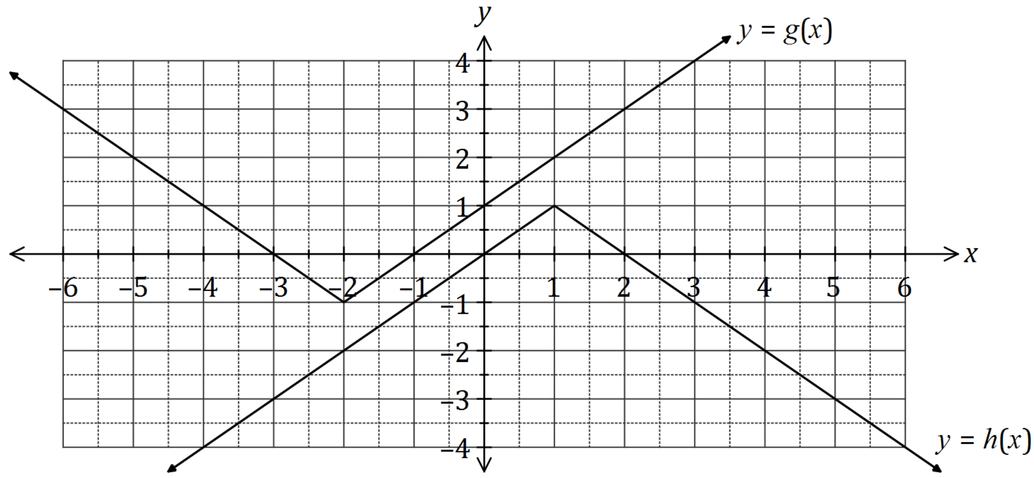
Solution Vector $\boldsymbol{n} = 4\boldsymbol{i} - 3\boldsymbol{j} - 12\boldsymbol{k}$ in perpendicular to the plane P and vector $\boldsymbol{d} = 3\boldsymbol{i} + \boldsymbol{k}$ is parallel to the line L. 1. $\begin{pmatrix} 4 \\ -3 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0$ ∴ line L is parallel to plane P. 2. $\begin{pmatrix} 4 \\ -3 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 8 - 12 - 12 = -16 \neq -12$ 3. Line L is parallel to plane P and a point on line L does not belong to plane P therefore line L dose not intersect with plane P.	
Marking key/mathematical behaviours	Marks
• Calculate the dot product of $\boldsymbol{n}$ and $\boldsymbol{d}$ • Select a point on the line and show that it does not belong to P • Write a correct conclusion	1 1 1

(d) Determine the distance between plane P and line L. (3 marks)

Solution	
1.Position vector of a point on the plane P: A(0, 0, 1); position vector of a point on the line L: B(1, 4, 1) $$\overrightarrow{AB} = \begin{pmatrix}1\\4\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\4\\0\end{pmatrix}$$ 2. Unit vector normal to the plane: $$\hat{\boldsymbol{n}} = \frac{1}{13}\begin{pmatrix}4\\-3\\-12\end{pmatrix}$$ 3. $\frac{1}{13}\begin{pmatrix}1\\4\\0\end{pmatrix} \cdot \begin{pmatrix}4\\-3\\-12\end{pmatrix} = \frac{-8}{13}$ ∴the distance between the line L and plane P is $\frac{8}{13} \approx$ $0.62\ units$	
Marking key/mathematical behaviours	Marks
• obtains a vector between a point on the line L and a point on the plane P • obtains a unit vector normal to the plane. • Obtain the dot product and state the solution.	1 1 1

Question 17 **(9 marks)**

(a) The graphs of the functions g and h are shown below.



Determine the value(s) of k if

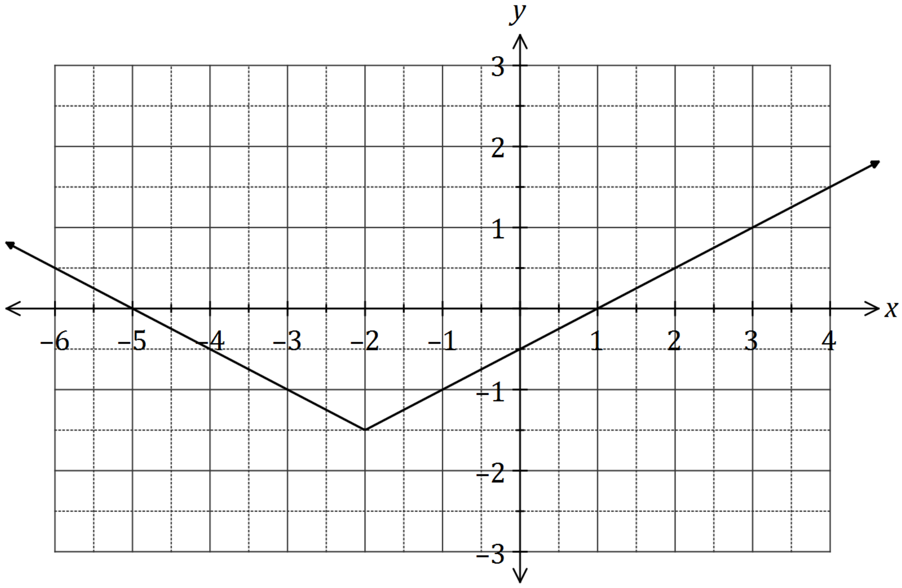
(i) $k = h \circ g(2)$. (1 mark)

Solution
$k = h(3) = -1$
Specific behaviours
✓ correct value

(ii) $g\left(h(k)\right) = 1$. (2 marks)

Solution
$g(x) = 1 \Rightarrow x = 0, -4$ $h(k) = 0 \Rightarrow k = 0, 2$ and $h(k) = -4 \Rightarrow k = -4, 6$ $k = -4, 0, 2, 6$
Specific behaviours
✓ indicates $h(k) = 0$ or -4 ✓ states all 4 values

(b) The graph of $f(x) = a|x - p| + q$ is shown below.



(i) Determine the value of the constants a, p and q. (3 marks)

Solution
Gradient: $a = \frac{1}{2}$ Vertical translation: $q = -1.5$ Horizontal translation: $p = -2$
Specific behaviours
✓✓✓ each value

(ii) If the equation $|f(x)| = mx + c$ has an infinite number of solutions, determine the values of the positive constants m and c. (3 marks)

Solution
For infinite solutions, we require straight line to be a part of $|f(x)|$. For m and c to be positive, must be part of $|f(x)| \text{where} - 5 \leq x \leq -2$ $m = \frac{1}{2}$ and $c = \frac{5}{2}$
Specific behaviours
✓ indicates must be one of four segments of $|f(x)|$ ✓ determines m ✓ determines c

Question 18 **(10 marks)**

The Cartesian equation of the sphere S is

$$x^2 - 6x + y^2 + z^2 + 10z = 2.$$

(a) Determine the radius and the coordinates of the centre of the sphere S. (3 marks)

Solution $x^2 - 6x + y^2 + z^2 + 10z = 2 \Leftrightarrow (x-3)^2 + y^2 + (z+5)^2 = 2 + 9 + 25 = 36 = 6^2$ So the radius is 6 and the centre C has coordinates $(3, 0, -5)$	
Marking key/mathematical behaviours	Marks
• completes the square • obtains correct radius • obtains correct coordinates of C	1 1 1

The vector equation of the line L is

$$\boldsymbol{r}(t) = (9 + 2t)\boldsymbol{i} - 2t\boldsymbol{j} + (1 + t)\boldsymbol{k}$$

(b) Does the line L intersect the sphere S and if so, where? (5 marks)

Solution Substituting $(x, y, z) = (9 + 2t, -2t, 1 + t)$ in the equation of the sphere gives $(9 + 2t)^2 - 6(9 + 2t) + (-2t)^2 + (1 + t)^2 + 10(1 + t) = 2,$ i.e. $38 + 36t + 9t^2 = 2$, i.e. $9(t + 2)^2 = 0$, i.e. $t = -2$ (or by calculator) so ℓ and $\mathcal{S}$ intersect so $(x, y, z) = (5, 4, -1)$ at the only point of intersection	
Marking key/mathematical behaviours	Marks
• substitutes correctly • sets up an equation for t • solves for t • deduces that the line and sphere intersect • solves for the coordinates x, y and z at the only point of intersection	1 1 1 1 1

(c) Explain why L is tangential to S. (2 marks)

Solution	
If P is the point of intersection $CP = 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$, The vector $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ is parallel to the line ℓ, And $(2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 4 - 8 + 4 = 0$ So the direction of the line is perpendicular to the radial vector CP. Since P is the point of intersection of ℓ and $\mathcal{S}$, ℓ must be tangential to $\mathcal{S}$ at P.	
Marking key/mathematical behaviours	Marks
• shows that ℓ is perpendicular to the radial vector at the point of intersection • argues that this implies the tangency property for ℓ	1 1

Question 19 **(6 marks)**

Determine, where possible, a unique solution for the following systems of equations. In each case, interpret the system of equations geometrically.

(a) $8x + y + z = 15$, $2x + y - z = 3$, and $x - y + 2z = 3$. (2 marks)

Solution
No unique solution, as infinite number of solutions exist. As planes clearly not parallel, then they represent three planes that intersect in a straight line. $(x = t, y = -5t + 9, z = 6 - 3t)$
Specific behaviours
✓ indicates infinite number of solutions ✓ indicates planes intersecting in a straight line

(b) $x + y - z = 0$, $x - y + 2z = 10$ and $3x - y + z = 16$. (2 marks)

Solution
Using CAS, $x = 4, y = -2, z = 2$ Three planes that intersect at the point $(4, -2, 2)$.
Specific behaviours
✓ solution ✓ interpretation

(c) $x + y = z + 2$, $x - y + z = 1$ and $x + z = y + 3$. (2 marks)

Solution
No solutions exist. Two parallel planes cut by the other plane. *(Last plane can be written $x - y + z = 3$ - parallel and non-intersecting with $x - y + z = 1$).*
Specific behaviours
✓ indicates no solutions ✓ indicates two parallel planes cut by third

Question 20 **(7 marks)**

Given the following system of equations for variables x, y, z.

$$\begin{cases} 2x - 3y + z = 4 \\ x - y + \boldsymbol{p}z = 3 \\ x - 2y + 2z = \boldsymbol{p^2} \end{cases}$$

(a) Use Gaussian elimination to show that the given system of equations always has at least one solution.

(5 marks)

Solution – Not Finished
Write given equations in augmented matrix form and performing Gaussian Elimination. $\left(\begin{array}{ccc\|c} 2 & -3 & 1 & 4 \\ 1 & -1 & p & 3 \\ 1 & -2 & 2 & p^2 \end{array}\right)$ $\left(\begin{array}{ccc\|c} 2 & -3 & 1 & 4 \\ 1 & -1 & p & 3 \\ 2 & -3 & p+2 & 3+p^2 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \to R3 + R2 \end{array}$ $\left(\begin{array}{ccc\|c} 2 & -3 & 1 & 4 \\ 1 & -1 & p & 3 \\ 0 & 0 & p+1 & p^2-1 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \to R3 - R1 \end{array}$ For $1 + p = 0 \Rightarrow p = -1$, this means $p^2 - 1 = 0 \implies (0 \quad 0 \quad 0\|0)$ ∴ for $p = -1$ system has infinite number of solutions. ∴ “No solutions” is not possible and there will always be at least one solution.
Specific behaviours
✓ write given equations in augmented matrix form ✓ perform Gaussian Elimination ✓ Identify the general condition for “No solutions” ✓ Show that this condition leads to system with infinitely many solutions ✓ Conclude that the system “No solution” is impossible

(b) Describe clearly the conditions required on $\boldsymbol{p}$ so that the system of equations has only one solution. (2 marks)

Solution – Not Finished
For there to be only one solution, we require: $(0 \quad 0 \quad a\|b)$ Were a is non-zero constant (b can be zero) ∴ $1 + p \neq 0 \Rightarrow p \neq -1$ ∴ system has unique solution for $p \neq -1$.
Specific behaviours
✓ Describe general condition for a unique solution. ✓ Write the solution.

End of questions

Additional working space

Question number: _________